Strongly gravitating empty spaces

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Abstract

We use various results concerning isometry groups of Riemannian and pseudo-Riemannian manifolds to prove that there are spaces on which differential structure can act as a source of gravitational force (Brans conjecture). The result is important for the analysis of the possible physical meaning of differential calculus. Possible astrophysical consequences are discussed.

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1 Introduction

The choice of mathematical model for spacetime has important physical significance. B. Riemann suggested that the geometry of space may be more than just a mathematical tool defining a stage for physical phenomena, and may in fact have profound physical meaning in its own right [1]. With the advent of general relativity theoreticians started to think of the spacetime as a differential manifold. Since then various assumptions about the spacetime topology and geometry have been discussed in the literature [2]. Until recently, the choice of differential structure of the spacetime manifold has been assumed to be trivial because most topological spaces used for modeling spacetime have natural differential structures and these differential structures where (wrongly) thought to be unique. Therefore the counterintuitive discovery of exotic $\mathbf{R}^4$'s following from the work of Freedman [3] and Donaldson [4] raised various discussions about the possible physical consequences of this discovery. Exotic $\mathbf{R}^4$'s are smooth (C^∞) four-manifolds which are homeomorphic to the Euclidean four-space $\mathbf{R}^4$ but not diffeomorphic to it. Exotic $\mathbf{R}^4$'s are unique to dimension four, see [5-11] for details. Later we have realized that exotic (nonunique) smooth structures are abundant in dimension four. For example it is sufficient to remove one point from a given four-manifold to obtain a manifold with exotic differential structures

[11]; every manifold of the form $M \times \mathbf{R}$, M being compact 3-manifold has infinitely many inequivalent differential structures. Such manifolds play important role in theoretical physics and astrophysics. Therefore the physical meaning of exotic smoothness must be thoroughly investigated. This is not an easy task: we only know few complicated coordinate descriptions [12] and most mathematicians believe that there is no finite atlas on an exotic $\mathbf{R}^4$. To our knowledge, only few physical examples have been discussed in the literature [2,6,7,13,14]. In this paper we would like to discuss some peculiarities that may happen while studying the theory of gravity on some exotic $\mathbf{R}^4$'s. The most important result is that on some topologically trivial spaces there exist only "complicated" solutions solutions to the Einstein equations. By this we mean that there may be no stationary cosmological model solutions and/or that empty space can gravitate. Such solutions are counterintuitive but we are aware of no physical principle that would require rejection of such spacetimes (besides common sense?).

2   General relativity on exotic $\mathbf{R}^4$'s with few symmetries

As it was written in the previous section, exotic $\mathbf{R}^4$'s are defined as four-manifolds that are homeomorphic to the fourdimensional Euclidean space $\mathbf{R}^4$ but not diffeomorphic to it. There are infinitely many of such manifolds (at

least a two parameter family of them) [5]. Note that exotic differential structures do not change the definition of the derivative. The essential difference is that the rings of real differentiable functions are different on nondiffeomorphic manifolds. In the case of exotic $\mathbf{R}^4$'s this means that there are some continuous functions $\mathbf{R}^4 \mapsto \mathbf{R}$ that are smooth on one exotic $\mathbf{R}^4$ and only continuous on another and vice versa [9]. To proceed we will recall several definitions. We will call a diffeomorphism $\phi : M \mapsto M$, where M is a (pseudo-)Riemannian manifold with metric tensor g an isometry if and only if it preserve g, $\phi^* g = g$ [15]. Such mappings form a group called the isometry group. We say that a smooth manifolds has few symmetries provided that for every choice of differentiable metric tensor, the isometry group is finite. Recently, L. R. Taylor managed to construct examples of exotic $\mathbf{R}^4$'s with few symmetries [16]. Among these there are examples with nontrivial but still finite isometry groups. Taylor's result, although concerning Riemannian structures, has profound consequences for the analysis of the possible role of differential structures in physics where Lorentz manifolds are commonly used. To show this let us define a (non-)proper actions of a group on manifolds as follows. Let G be a locally compact topological group acting on a metric space X. We say that G acts properly on X if and only if for all compact subsets $Y \subset X$, the set $\{g \in G : gY \cap Y \neq \emptyset\}$ is also compact. Restating this we say that G acts nonproperly on X if and only if there exist

sequences $x_n \to x$ in X and $g_n \to \infty$ in G, such that $g_n x_n$ converges in X. Here $g_n \to \infty$ means that the sequence g_n has no convergent subsequence in the compact open topology on the set of all isometries [15 p. 202]. Note that for many manifolds a proper G action is topologically impossible and on the other hand a nonproper G action on a Lorentz (or pseudo-Riemannian) manifolds is for all but a few groups also impossible [17, 18]. Our discussion would strongly depend on the later fact and on the theorems proved by N. Kowalsky [18]. First of all let us quote [18]:

Theorem 1 *Let G be Lie transformation group of a differentiable manifold X. If G acts properly on X, then G preserves a Riemannian metric on X. The converse is true if G is closed in Diff(X).*

As a special case we have:

Theorem 2 *Let G and X be as above, and in addition assume G connected. If G acts properly on X preserving a time-orientable Lorentz metric, then G preserves a Riemannian metric and an everywhere nonzero vector field on X.*

If we combine these theorems with the Taylor's results we immediately get:

Theorem 3 *Let G be a Lie transformation group acting properly on an exotic $\mathbf{R}^4$ with few symmetries and preserving a time-orientable Lorentz metric. Then G is finite.*

Further, due to N. Kowalsky, we also have [18]:

Theorem 4 *Let G be a connected noncompact simple Lie group with finite center. Assume that G is not locally isomorphic to SO(n, 1) or SO(n,2). If G acts nontrivially on a manifold X preserving a Lorentz metric, then G actually acts properly on X.*

and

Theorem 5 *If G acts nonproperly and nontrivially on X, then G must be locally isomorphic to SO(n,1) or SO(n,2) for some n.*

The general nonproper actions of Lie groups locally isomorphic to SO(n,1) or SO(n,2) would be discussed in ref. [19]. In many cases it is possible to describe the cover $\tilde{X}$ up to Lorentz isometry.

Now, suppose we are given an exotic $\mathbf{R}_\theta^4$ with few symmetries. Given any boundary conditions, we can try to solve the Einstein equations on $\mathbf{R}_\theta^4$. Suppose we have found some solution to the Einstein equations on $\mathbf{R}_\theta^4$. Whatever the boundary conditions be we would face one of the two following situations.

- The isometry group G of the solution acts properly on $\mathbf{R}_\theta^4$. Then according to Theorem 3 G is finite. There is no nontrivial Killing vector field and the solution cannot be stationary [19]. The gravitation is

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quite "complicated" and even empty spaces do evolve. Note that this conclusion is valid for any open subspace of $\mathbf{R}_\theta^4$. This means that this phenomenon cannot be localized on such spacetimes.

- The isometry group G of the solution acts nonproperly on $\mathbf{R}_\theta^4$. Then G is locally isomorphic to SO(n,1) or SO(n,2). But the nonproper action of G on $\mathbf{R}_\theta^4$ means that there are points infinitely close together in $\mathbf{R}_\theta^4$ $(x_n \to x)$ such that arbitrary large different isometries $(g_n \to \infty)$ in G maps them into infinitely close points in $\mathbf{R}_\theta^4$ $(g_n x_n \to y \in \mathbf{R}_\theta^4)$. There must exists quite strong gravity centers to force such convergence (even in empty spacetimes). Such spacetimes are unlikely to be stationary.

We see that in both cases Einstein gravity is quite nontrivial even in the absence of matter. Let us recall that if a spacetime has a Killing vector field ζ^a, then every covering manifolds admit appropriate Killing vector field ζ'^a such that it is projected onto ζ^a by the differential of the covering map. This means that discussed above properties are "projected" on any space that has exotic $\mathbf{R}^4$ with few symmetries as a covering manifold eg quotient manifolds obtained by a smooth action of some finite group. Note that we have proven a weaker form of the Brans conjecture [7]: there are four-manifolds (spacetimes) on which differential structures can act as a source of gravitational force just as ordinary matter does.

3 Conclusions

The existence of topologically trivial spacetimes that admit only "nontrivial" solutions to the Einstein equations is very surprising. Such phenomenon might be also possible for other four-manifolds admitting exotic differential structures enumerated in the Introduction. The first reaction is to reject them as being unphysical mathematical curiosities. But this conclusion might be erroneous [6-8, 13]. Besides the arguments put forward by Brans [6-8] and Asselmeyer [13] we would like to add the following. Suppose that space-time is only a secondary entity emerging as a result of interactions between physical (matter) fields. If we use the A. Connes' noncommutative geometry formalism do describe Nature then Dirac operators and their spectra define the spacetime structure [22, 23]. There are known examples of differential structures that are distinguished by spectra of the Dirac operators [23-25]. This suggest that fundamental interactions of matter are "responsible" for the selection of the differential structure and might not have "chosen" the simplest structure of the spacetime manifold. Although it is unlikely that the spectrum of the Dirac operator alone would allow to distinguish the differential structure [26] in a general case, it seems to be reasonable to conjecture that physically equivalent spacetimes must be isospectral [2] and we might hope that this would solve the problem with the plethora of exotic differ-

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ential structures. If Nature have not used exotic smoothness we physicists should find out why only one of the existing differential structures has been preferred. Does it mean that the differential calculus, although very powerful, is not necessary (or sufficient) for the description of the laws of physics? It might not be easy to find any answer to these questions.

Let us conclude by saying that if exotic smoothness has anything to do with the physical world it may be a source/ explanation of various astrophysical and cosmological phenomena. Dark matter and vacuum energy substitutes and attracting centers are the most obvious among them [27-29]. "Exoticness" of the spacetime might be responsible for the recently discovered anomalies in the large redshift supernovae properties. The process of "elimination" of exotic differential structures might also result in the emergence time [30, 31] or spacetime signature.

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